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On $L_{\infty\kappa}$-free Boolean algebras*

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Abstract

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We study $L_{\infty\kappa}$-freeness in the variety of Boolean algebras. It is shown that some of the theorems on $L_{\infty\kappa}$-free algebras which are known to hold in varieties such as groups, abelian groups etc. are also true for Boolean algebras. But we also investigate properties such as the ccc of $L_{\infty\omega_1}$-free Boolean algebras which have no counterpart in the varieties above.

0. Introduction

For a cardinal κ and a fixed variety $\mathscr{V}$ in a countable language, an algebra A in $\mathscr{V}$ is said to be $L_{\infty\kappa}$-free if A is $L_{\infty\kappa}$-equivalent to a free algebra in $\mathscr{V}$. $L_{\infty\kappa}$-free algebras in various varieties have been investigated by several authors (see e.g. [3, 4, 5, 16]). In this paper we shall study the case of the variety of Boolean algebras. The most distinctive property of Boolean algebras in this connection is that every atomless (i.e., $L_{\infty\omega}$-free) Boolean algebra of size ω_1 has the property that the set of all countable free subalgebras is closed unbounded in the set of all countable subalgebras. Another peculiarity is that subalgebras of free algebras are not necessarily free. While the first one is rather typical for Boolean algebras, the second property occurs also in some other known varieties.

An $L_{\infty\kappa}$-free Boolean algebra is almost free in some sense. We shall consider yet another notion of almost freeness: a Boolean algebra B is said to be κ-free if the set of free subalgebras of B of size $<\kappa$ contains a club set in $[B]^{<\kappa}$.

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We shall treat the following questions:

1. For which cardinals κ does there exist an $L_{\infty\kappa}$-free non-free Boolean algebra?
2. Are $L_{\infty\kappa}$-free Boolean algebras κ-free?
3. What kind of Boolean algebras can be embedded into an $L_{\infty\omega_1}$-free Boolean algebra?
4. How can we characterize Boolean algebras which can be forced to be free?

One of the problems which has been considered in the literature is to determine, for a given variety $\mathscr{V}$ in a countable language, the class $\mathscr{K}$ of infinite cardinals κ for which there exists an $L_{\infty\kappa}$-free non-free algebra in $\mathscr{V}$ of size κ. Shelah showed that this class does not contain any singular cardinal (Shelah's Singular Compactness Theorem, see [16, 7]). Eklof and Mekler [4] showed that, assuming $V = L$, if $\mathscr{K}$ contains a non-weakly compact regular uncountable cardinal, then it contains all regular non-weakly compact uncountable cardinals. So under $V = L$, $\mathscr{K}$ can be determined almost uniquely. What can we say about this class in ZFC without any further set-theoretic assumptions? We consider this problem in the case of Boolean algebras.

It turns out that the notion of projectivity works quite well in the study of the questions above. The fundamental results about projectivity which we use repeatedly are the theorems by Ščepin, Haydon and the second author, cited here as Theorem 1.3 and Theorem 1.4 in the next section. By virtue of Theorem 1.4, every $L_{\infty\kappa}$-free projective Boolean algebra of size κ is free. So in order to construct a Boolean algebra which is $L_{\infty\kappa}$-free but not free, we have to construct an $L_{\infty\kappa}$-free Boolean algebra which is not projective.

In case of $\kappa = \omega_1$ or ω_2 we can actually construct (already in ZFC) $L_{\infty\kappa}$-free Boolean algebras which are not projective (Corollary 2.5). We also show that every $L_{\infty\omega_2}$-free Boolean algebra of size ω_2 is ω_2-free (Theorem 2.7).

In dealing with the third question, we consider the following 'stability' property of a structure A:

(Stab) For every countable substructure A' of A, only countably many types in quantifier-free formulas over A' are realized in A.

Since free algebras in any variety $\mathscr{V}$ satisfy the condition (Stab) and (Stab) can be formulated as an $L_{\infty\omega_1}$-sentence, every $L_{\infty\omega_1}$-free algebra in $\mathscr{V}$ and its subalgebras also satisfy (Stab). On the other hand we still do not know if e.g. every Boolean algebra with the property (Stab) can be embedded into an $L_{\infty\omega_1}$-free one. As a partial answer to this question, we shall show in Theorem 3.5 that a pseudo-tree algebra B can be embedded into an $L_{\infty\omega_1}$-free Boolean algebra if and only if B satisfies the condition (Stab).

Every atomless Boolean algebra can be forced to be free if simply its cardinality is collapsed to be countable. So our fourth problem makes sense when e.g. the considered generic extensions preserve cardinality of the Boolean algebra. Let us call a Boolean algebra B κ-potentially free if it is isomorphic to a free Boolean algebra in the Boolean extension $V^{(C)}$ of the universe V for some $(<\kappa, \infty)$-distributive complete Boolean algebra C. In Section 4 we give a characterization of κ-potentially free Boolean algebras of size κ. The invariant $\mathrm{Gd}(B)$ — an element of $\mathcal{P}(\kappa)$ modulo $\mathrm{Cub}(\kappa)$ (= the closed unbounded filter on κ) — for a Boolean algebra B of size κ defined in the next section is used in the characterization. We shall show that B is projective if and only if $\mathrm{Gd}(B)$ is the greatest element of $\mathcal{P}(\kappa)/\mathrm{Cub}(\kappa)$ (Proposition 2.2). Using this fact we obtain a Boolean algebra version of a theorem in [1] (Theorem 4.2). Combining this with the construction of $L_{\infty\kappa}$-free Boolean algebras introduced in Section 2 we can show that e.g. under $\mathrm{V} = \mathrm{L}$ there are maximally many (i.e., 2^{κ}) pairwise non-isomorphic κ-potentially free Boolean algebras of size κ, for each regular non-weakly compact uncountable cardinal κ (Corollary 4.4).

1. Preliminaries

In this section we give basic definitions, notation and results which are needed in this paper. The reader is assumed to be familiar with the basic facts about Boolean algebras found in [9]. Details on notions of relative completeness and projectivity can be found in [10]. For basic facts about set theory, we refer to [8, 13].

We use the letters κ, λ for infinite cardinals; the letters α, β, δ for ordinals; the letters A, B, C for (in most cases infinite) Boolean algebras. For a set X, we denote by $|X|$ the cardinality of X and by $\mathcal{P}(X)$ the Boolean algebra of all subsets of X. 2 is the two-element Boolean algebra. $A \leqslant B$ means that A is a subalgebra of B.

If $X \subset A$, then $\langle X \rangle$ is the subalgebra of A generated by X. B is countably generated over A if $A \leqslant B$ and $B = \langle A \cup X \rangle$ for some countable subset X of B. B is a simple extension of A if $B = \langle A \cup \{x\} \rangle$ for some $x \in B$. $\mathrm{Sub}_{<\kappa}(A)$ is the set of subalgebras of A of size $<\kappa$.

For any set X, let $\mathrm{Fr}\, X$ be the free Boolean algebra over X. We assume without loss of generality that $X \subset \mathrm{Fr}\, X$ and that $X \subset Y$ implies $\mathrm{Fr}\, X \leqslant \mathrm{Fr}\, Y$. We write $A \mid B$ if $B = A \oplus F$ for some free Boolean algebra F where we assume A and F to be independent subalgebras of their free product $A \oplus F$.

For a regular cardinal κ, we say that B is κ-free if $\{C \in \mathrm{Sub}_{<\kappa}(B)\colon C \text{ is free}\}$ is closed unbounded in $\mathrm{Sub}_{<\kappa}(B)$. Note that a Boolean algebra of size κ is κ-free if and only if there exists a filtration $(B_\alpha)_{\alpha<\kappa}$ of B (see below for this notion) such that B_α is free for every $\alpha < \kappa$.

For a cardinal κ, $L_{\infty\kappa}$ is the infinitary logic which allows conjunction and disjunction of arbitrary set of formulas as well as quantification over any block of variables of length $<\kappa$. B is $L_{\infty\kappa}$-free if B is $L_{\infty\kappa}$-equivalent to some free Boolean algebra. We use the following characterization of $L_{\infty\kappa}$-freeness.

Theorem 1.1. *Let κ be regular. The following conditions are equivalent.*

(1) *B is $L_{\infty\kappa}$-free.*

(2) *There is a cofinal subset $\mathcal{S}$ of $\mathrm{Sub}_{<\kappa}(B)$ such that every $S \in \mathcal{S}$ is free and for any finite $\mathcal{S}' \subset \mathcal{S}$ there exists $S \in \mathcal{S}$ such that $A \mid S$ for every $A \in \mathcal{S}'$.*

(3) (Kueker [12]) *There is a cofinal subset $\mathcal{S}$ of $\mathrm{Sub}_{<\kappa}(B)$ such that every $S \in \mathcal{S}$ is free and for any $\mathcal{S}' \subset \mathcal{S}$ with $|\mathcal{S}'| < \kappa$ there exists $S \in \mathcal{S}$ such that $A \mid S$ for every $A \in \mathcal{S}'$.* (We shall call an $\mathcal{S} \subseteq \mathrm{Sub}_{<\kappa}(B)$ with this property *a Kueker system for B*.)

Proof. (2) is just a reformulation of the back-and-forth argument. A simple proof of (3) can be found in [3]. $\square$

Let A be a subalgebra of B. We say that A is relatively complete in B ($A \leqslant_{\mathrm{rc}} B$) if for every $b \in B$ there is a largest element a of A satisfying $a \leqslant b$. a as above is said to be the projection of b in A and denoted by $\mathrm{pr}^B_A(b)$. We write $A \leqslant_{\mathrm{rc}\omega} B$ when $A \leqslant_{\mathrm{rc}} B$ and B is countably generated over A. B is a projective extension of A ($A \leqslant_{\mathrm{proj}} B$) if there are a free Boolean algebra F and homomorphisms $e: B \to A \oplus F$, $q: A \oplus F \to B$ such that $q \circ e = \mathrm{id}_B$ and $e \restriction A = q \restriction A = \mathrm{id}_A$. If B is a projective extension of 2, we say simply that B is projective. We define the weight of B over A by

$$\mathrm{wt}(B/A) = \min\{|X| : X \subset B,\ \langle A \cup X \rangle = B\};$$

we write $\mathrm{wt}(B)$ if $A = 2$, i.e., $\mathrm{wt}(B) = |B|$. A set $\mathcal{S}$ of subalgebras of B is called a skeleton for B over A if it satisfies the following conditions:

(1) $A \in \mathcal{S}$.

(2) $C \in \mathcal{S}$ implies $A \leqslant C \leqslant_{\mathrm{rc}} B$.

(3) If $\mathcal{S}' \subset \mathcal{S}$ is a non-empty chain under set-theoretic inclusion, then $\bigcup \mathcal{S}' \in \mathcal{S}$.

(4) For $C \in \mathcal{S}$ and X a countable subset of B, there is $C' \in \mathcal{S}$ such that $C \cup X \subset C'$ and $C \leqslant_{\mathrm{rc}\omega} C'$.

The proof of the following Lemma 1.2 and Theorems 1.3, 1.4 can be found in [10].

Lemma 1.2. (1) *$A \mid B$ implies $A \leqslant_{\mathrm{proj}} B$; $A \leqslant_{\mathrm{proj}} B$ implies $A \leqslant_{\mathrm{rc}} B$.*

(2) *If $|A| \leqslant \omega_1$* and *$A \leqslant_{\mathrm{rc}} F$ for some free Boolean algebra F, then A is projective.*

(3) *If $A \leqslant_{\mathrm{proj}} B$, then A is a retract of B, i.e., there is a homomorphism $f: B \to A$ such that $f \circ i = \mathrm{id}_A$ where i is the inclusion map of A into B.* $\square$

Theorem 1.3. *Let A be a subalgebra of a Boolean algebra B. The following conditions are equivalent:*

(1) *B is a projective extension of A.*

(2) (Ščepin) *There is a skeleton for B over A.*

(3) (Haydon) *B is the union of a continuous chain $(B_\alpha)_{\alpha<\rho}$ (for some ordinal ρ) of subalgebras such that $B_0 = A$, B_α is relatively complete in $B_{\alpha+1}$ and $B_{\alpha+1}$ is countably generated over B_α.*

(4) (Koppelberg) *B is the union of a continuous chain $(B_\alpha)_{\alpha<\rho}$ (for some ordinal ρ) of subalgebras such that $B_0 = A$, B_α is relatively complete in $B_{\alpha+1}$ and $B_{\alpha+1}$ is a simple extension of B_α.* □

Theorem 1.4 (Ščepin). *Let B be a projective Boolean algebra and $\kappa = |B| \geq \omega$. Assume that every ultrafilter p of B has character κ, i.e., every set of generators of p has size κ. Then B is a free Boolean algebra on κ generators.* □

Lemma 1.5. (1) *Let $A \leq_{\text{proj}} B$ and $\kappa = \text{wt}(B/A) + \omega$. Assume that every ultrafilter p of B has character κ over A, i.e., for every $S \subseteq p$ such that $(p \cap A) \cup S$ generates p, we have $|S| = \kappa$. Then $A \mid B$. In particular $A \leq_{\text{proj}} B$ if and only if $A \mid B \oplus \text{Fr}\,\kappa$ for $\kappa = \text{wt}(B/A) + \omega$.*

(2) *$C \leq B \leq_{\text{proj}} A$ and $C \leq_{\text{proj}} A$ imply $C \leq_{\text{proj}} B$.*

Proof. (1) The proof is similar to that of Theorem 1.4 as stated in [10]. Let $\mathcal{S}$ be a skeleton for B over A. For any $C \in \mathcal{S}$ with $\text{wt}(C/A) < \kappa$ and any $x \in B$, there exists $z \in B$ such that z is independent over C and $x \in \langle C \cup \{z\}\rangle$. Hence we can find a continuously increasing chain $(C_\alpha)_{\alpha<\kappa}$ in $\mathcal{S}$ such that $C_0 = A$, $\bigcup_{\alpha<\kappa} C_\alpha = B$ and $C_\alpha \mid C_{\alpha+1}$ for every $\alpha < \kappa$. Thus $A \mid B$.

(2) By Lemma 1.2(3) and by the definition of projectivity, there are homomorphisms

$$B \underset{f}{\overset{i}{\rightleftarrows}} A \underset{q}{\overset{e}{\rightleftarrows}} C \oplus F$$

for some free F such that i is the inclusion map of B into A, $f \circ i = \text{id}_B$, $q \circ e = \text{id}_A$ and $e \restriction C = q \restriction C = \text{id}_C$. Then $(f \circ q) \circ (e \circ i) = \text{id}_B$ and $(e \circ i) \restriction C = (f \circ q) \restriction C = \text{id}_C$. Hence $C \leq_{\text{proj}} B$. □

A sequence $(B_\alpha)_{\alpha<\kappa}$ of subalgebras of B is said to be a filtration of B over A if

(1) $A = B_0 \subset B_\alpha \subset B_{\alpha+1} \subset B$ for every $\alpha < \kappa$;

(2) $B_\delta = \bigcup_{\alpha<\delta} B_\alpha$ for every limit ordinal $\delta < \kappa$;

(3) $B = \bigcup_{\alpha<\kappa} B_\alpha$;

(4) $\text{wt}(B_\alpha/A) < \text{wt}(B/A) = \kappa$.

$(B_\alpha)_{\alpha<\kappa}$ is a filtration of B if it is a filtration of B over some B_0 such that $|B_0| < |B|$. If $(B_\alpha)_{\alpha<\kappa}$, $(B'_\alpha)_{\alpha<\kappa}$ are filtrations of B over A and $\kappa = \text{wt}(B/A)$ is regular uncountable, then there is a club subset C of κ such that $B_\alpha = B'_\alpha$ for every $\alpha \in C$. We say that a sequence $(B_\alpha)_{\alpha<\kappa}$ is a continuous chain of B, if it satisfies (1)–(3) above.

We call the filtration $(\text{Fr}\,\alpha)_{\alpha<\kappa}$ of $\text{Fr}\,\kappa$ the canonical free-filtration of $\text{Fr}\,\kappa$.

If B is a projective extension of A, then there is a filtration $(B_\alpha)_{\alpha<\kappa}$ of B over A which satisfies the condition (3) of Theorem 1.3. We call such a filtration a projective filtration of B over A. By Kueker's criterion in Theorem 1.1, if B is an $L_{\infty\kappa}$-free Boolean algebra of size κ and κ is regular, there is a filtration $(B_\alpha)_{\alpha<\kappa}$ of B over 2 such that for every $\alpha<\beta<\kappa$, $B_{\alpha+1}\cong \mathrm{Fr}\,|\alpha+\omega|$ and $B_{\alpha+1} \mid B_{\beta+1}$. We call such a filtration an $L_{\infty\kappa}$-filtration of B. We call B a weakly projective extension of A and write $A \leqslant_{\mathrm{wproj}} B$ if there exists an increasing chain $(B_\alpha)_{\alpha<\kappa}$ such that

(1) $\quad B = \bigcup_{\alpha<\kappa} B_\alpha;$

(2) $\quad B_\delta = \bigcup_{\alpha<\delta} B_\alpha \quad$ for all $\delta<\kappa$ with $\mathrm{cof}(\delta)=\omega$;

(3) $\quad \mathrm{wt}(B_\alpha/A) < \mathrm{wt}(B/A) = \kappa \quad$ for all $\alpha<\kappa$;

(4) $\quad A \leqslant_{\mathrm{proj}} B_\alpha \quad$ for all $\alpha<\kappa$.

If $(B_\alpha)_{\alpha<\kappa}$ satisfies (1)–(4) above we say that $(B_\alpha)_{\alpha<\kappa}$ witnesses $A \leqslant_{\mathrm{wproj}} B$.

Lemma 1.6. (1) *$A \leqslant_{\mathrm{wproj}} B$ implies $A \leqslant_{\mathrm{rc}} B$.*

(2) *$A \leqslant_{\mathrm{rc}} B$ and $\mathrm{wt}(B/A) \leqslant \omega_1$ imply $A \leqslant_{\mathrm{wproj}} B$.*

(3) *Let $(B_\alpha)_{\alpha<\kappa}$ be a filtration of B over A. If $A \leqslant_{\mathrm{proj}} B_\alpha$ for every $\alpha<\kappa$, then $A \leqslant_{\mathrm{wproj}} B$. In particular, $A \leqslant_{\mathrm{proj}} B$ implies $A \leqslant_{\mathrm{wproj}} B$.*

(4) *Let κ be regular uncountable, B an $L_{\infty\kappa}$-free Boolean algebra of size κ and $(B_\alpha)_{\alpha<\kappa}$ an $L_{\infty\kappa}$-filtration of B. Then $B_{\alpha+1} \leqslant_{\mathrm{wproj}} B$ for every $\alpha<\kappa$.*

Proof. (1) By Lemma 1.2(1).

(2) Let $(B_\alpha)_{\alpha<\kappa}$, $\kappa \leqslant \omega_1$ be any filtration of B over A. Then by Theorem 1.3, $(B_\alpha)_{\alpha<\kappa}$ witnesses $A \leqslant_{\mathrm{wproj}} B$.

(3) and (4) follow immediately from the definition. □

Proposition 1.7. *Suppose that $C \leqslant A \leqslant_{\mathrm{wproj}} B$, $\mathrm{wt}(A/C) < \mathrm{wt}(B/A) = \kappa$ and $\mathrm{cof}(\kappa) \geqslant \omega_1$. Then $C \leqslant_{\mathrm{wproj}} B$ if and only if $C \leqslant_{\mathrm{proj}} A$.*

Proof. Suppose that $C \leqslant_{\mathrm{wproj}} B$. Let $(A_\alpha)_{\alpha<\kappa}$ and $(C_\beta)_{\beta<\kappa}$ be witnesses of $A \leqslant_{\mathrm{wproj}} B$ and $C \leqslant_{\mathrm{wproj}} B$. Since $\mathrm{wt}(A/C) < \mathrm{wt}(B/C) = \mathrm{wt}(B/A) = \kappa$ and $\mathrm{cof}(\kappa) \geqslant \omega_1$, there exist $\alpha_0 < \alpha_1 < \cdots < \kappa$ and $\beta_0 < \beta_1 < \cdots < \kappa$ such that

$$A \subset A_{\alpha_0} \subset C_{\beta_0} \subset \cdots \subset A_{\alpha_n} \subset C_{\beta_n} \cdots \subset B.$$

Put $\bar{A} = \bigcup_{n<\omega} A_{\alpha_n} = \bigcup_{n<\omega} C_{\beta_n}$. Then $A \leqslant_{\mathrm{proj}} \bar{A}$ and $C \leqslant_{\mathrm{proj}} \bar{A}$. Hence $C \leqslant_{\mathrm{proj}} A$ by Lemma 1.5(2). Conversely if $C \leqslant_{\mathrm{proj}} A$, then, since $\mathrm{wt}(A/C) < \mathrm{wt}(B/A)$, any sequence $(A_\alpha)_{\alpha<\kappa}$ witnessing $A \leqslant_{\mathrm{wproj}} B$ also witnesses $C \leqslant_{\mathrm{wproj}} B$. □

2. $L_{\infty\kappa}$-free Boolean algebras

In this section we give a construction (in ZFC) of $L_{\infty\kappa}$-free non-free Boolean algebras for $\kappa = \omega_1$ or ω_2. Our method is similar to that of strongly κ-free non-free groups in [5]. For any infinite cardinal λ let $F(\lambda)$ be the following statement:

> There are free Boolean algebras A and B on λ generators such that $A \leqslant B$, $A \nleqslant_{\text{proj}} B$ and for any $C \mid A$ with $|C| < \lambda$ we have that $C \mid B$.

Lemma 2.1. *Let λ be a regular cardinal. Suppose that $F(\lambda)$ holds. For every free Boolean algebras A and $(A_\alpha)_{\alpha<\lambda}$ such that*

(1) *$(A_\alpha)_{\alpha<\lambda}$ is a strictly increasing continuous chain of subalgebras of A,*
(2) *$A = \bigcup_{\alpha<\lambda} A_\alpha$,*
(3) *$A_\alpha \mid A_\beta \mid A$ for $\alpha < \beta < \lambda$,*

there exists a free Boolean algebra B such that $A \leqslant B$, $A \nleqslant_{\text{proj}} B$, $|A| = |B|$ and $A_\alpha \mid B$ for every $\alpha < \lambda$.

Proof. Let $(A_\alpha)_{\alpha<\lambda}$ and A be as above. Let $(C_\alpha)_{\alpha<\lambda}$ be the canonical free-filtration of $\operatorname{Fr}\lambda$. By $F(\lambda)$ there is C such that $\operatorname{Fr}\lambda \nleqslant_{\text{proj}} C$ but $C_\alpha \mid C$ for every $\alpha < \lambda$. Let D_α, $\alpha < \lambda$ and D be defined by:

$$D_0 = \operatorname{Fr}(|A_0|),$$
$$D_{\alpha+1} = \operatorname{Fr}(|A_\alpha|) \oplus \operatorname{Fr}(\operatorname{wt}(A_{\alpha+1}/A_\alpha)),$$
$$D_\delta = \bigcup_{\alpha<\delta} D_\alpha \quad \text{if } \delta \text{ is limit},$$
$$D = \bigcup_{\alpha<\lambda} D_\alpha.$$

Let $B_\alpha = C_\alpha \oplus D_\alpha$ and $B = C \oplus D$. Since $C_\alpha \mid C$ and $D_\alpha \mid D$, we have that $B_\alpha \mid C \oplus D$. On the other hand, we have

$$B' = \bigcup_{\alpha<\lambda} B_\alpha = \operatorname{Fr}\lambda \oplus D \nleqslant_{\text{proj}} C \oplus D = B.$$

Identifying A and $(A_\alpha)_{\alpha<\lambda}$ with B' and $(B_\alpha)_{\alpha<\lambda}$ respectively, this B will do. $\square$

For regular uncountable κ and a filtration $(B_\alpha)_{\alpha<\kappa}$ of B over A let

$$\operatorname{Gd}((B_\alpha)_{\alpha<\kappa}) = \{\alpha < \kappa\colon A \leqslant_{\text{proj}} B_\alpha \leqslant_{\text{wproj}} B\}.$$

We define $\operatorname{Gd}(B/A)$ to be the equivalence class $\tilde{M}$ of $M = \operatorname{Gd}((B_\alpha)_{\alpha<\kappa})$ modulo $\operatorname{Cub}(\kappa)$ for some filtration $(B_\alpha)_{\alpha<\kappa}$ of B over A. We note that $\operatorname{Gd}(B/A)$ does not depend on the choice of the filtration. We write simply $\operatorname{Gd}(B)$ if $A = 2$.

Proposition 2.2. *Suppose that $A \leqslant B$ and $\kappa = \operatorname{wt}(B/A)$ is regular uncountable. Then $A \leqslant_{\text{proj}} B$ if and only if $\operatorname{Gd}(B/A) = \tilde{\kappa}$, the greatest element of $\mathcal{P}(\kappa)/\operatorname{Cub}(\kappa)$.*

Proof. Suppose that $A \leqslant_{\text{proj}} B$. Let $(B_\alpha)_{\alpha<\kappa}$ be a projective filtration of B over A. Then $\text{Gd}((B_\alpha)_{\alpha<\kappa}) = \kappa$. Hence we have $\text{Gd}(B/A) = \tilde{\kappa}$. Conversely, suppose that $\text{Gd}(B/A) = \tilde{\kappa}$. Let $(B_\alpha)_{\alpha<\kappa}$ be a filtration of B over A. $\text{Gd}((B_\alpha)_{\alpha<\kappa})$ contains a club subset of κ. Hence there is a filtration $(B'_\alpha)_{\alpha<\kappa}$ of B over A such that

$$A \leqslant_{\text{proj}} B'_\alpha \leqslant_{\text{wproj}} B \quad \text{for every } \alpha < \kappa.$$

By Proposition 1.7,

$$B'_\alpha \leqslant_{\text{proj}} B'_\beta \quad \text{for every } \alpha < \beta < \kappa.$$

Hence by Theorem 1.3 we obtain $A \leqslant_{\text{proj}} B$. $\square$

The following proposition corresponds to Theorem 2.3 in [5].

Proposition 2.3. *Let κ be a regular uncountable cardinal and S be a non-reflecting stationary subset of κ such that for every $\alpha \in S$, α is a limit ordinal and $F(\text{cof}(\alpha))$ holds. Then there is an $L_{\infty\kappa}$-free Boolean algebra A of size κ such that $\text{Gd}(A) = (\kappa \setminus S)^{\sim}$. In particular A is not free.*

Proof. We define a continuously increasing chain $(A_\alpha)_{\alpha<\kappa}$ such that

(1) A_α is a free Boolean algebra of size $|\alpha + \omega|$ for every $\alpha < \kappa$,

(2) $A_{\alpha+1} \mid A_{\beta+1}$ for every $\alpha < \beta < \kappa$,

(3) $\{\alpha < \kappa : A_\alpha \not\leqslant_{\text{wproj}} A\} = S$,

and put $A = \bigcup_{\alpha<\kappa} A_\alpha$. Then A is $L_{\infty\kappa}$-free by (1), (2) and Theorem 1.1, but non-projective by (3) and Proposition 2.2, hence non-free.

For $\alpha < \kappa$, we define A_α inductively such that

$$A_\alpha \cong \text{Fr}(|\alpha + \omega|)$$

and

$$(*)_\alpha \qquad \forall \beta < \alpha\, [(\beta \notin S \Rightarrow A_\beta \mid A_\alpha) \wedge (\beta \in S \Rightarrow A_\beta \not\leqslant_{\text{proj}} A_\alpha)].$$

Suppose that A_β, $\beta < \alpha$ are already defined.

Case 1: *α is a limit ordinal.* Let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Since $\alpha \cap S$ is not stationary, there is a club $X \subset \alpha$ such that

$$\forall \beta, \beta' \in X\, [\beta < \beta' \Rightarrow A_\beta \mid A_{\beta'}].$$

So $A_\alpha = \bigcup_{\beta \in X} A_\beta$ and $A_\alpha \cong \text{Fr}(|\alpha|)$.

Case 2: *$\alpha = \gamma + 1$ and $\gamma \notin S$.* Let $A_\alpha = A_\gamma \oplus \text{Fr}(|\gamma + \omega|)$.

Case 3: *$\alpha = \gamma + 1$ and $\gamma \in S$.* Let $\lambda = \text{cof}(\gamma)$ and $(\beta_\nu)_{\nu<\lambda}$ be a continuously increasing cofinal sequence in γ such that β_ν is not in S for every $\nu < \lambda$. Since $F(\lambda)$ holds, we can apply Lemma 2.1 to A_λ and $(A_{\beta_\nu})_{\nu<\lambda}$. So there is a free Boolean algebra B such that $A_\gamma \not\leqslant_{\text{proj}} B$ and $A_{\beta_\nu} \mid B$ for $\nu < \lambda$. Let $A_\alpha = B$.

In each of these cases, by Lemma 1.5(2), it is clear that $(*)_\alpha$ holds. $\square$

Theorem 2.4. *$F(\omega)$ and $F(\omega_1)$ hold.*

Proof. There are countable free Boolean algebras A, B such that $A \leqslant B$ but $A \not\leqslant_{\mathrm{rc}} B$ (hence $A \not\leqslant_{\mathrm{proj}} B$). For every finite Boolean algebra C and countable free Boolean algebra D such that $C \leqslant D$, we have that $C \mid D$. Hence $F(\omega)$ holds.

In order to show $F(\omega_1)$ we need the following observation due to Ščepin [15] which was originally formulated in a topological setting: For any Boolean algebra A we think of $A \oplus A$ as being generated by two independent subalgebras A_1 and A_2 isomorphic to A. For each Boolean algebra A and for $i = 1, 2$ let us fix isomorphisms

$$\varepsilon_i^A : A \to A_i.$$

Let

$$FA = A \oplus A$$

and for each Boolean homomorphism $f : A \to B$, let

$$Ff : FA \to FB$$

be the unique homorphism satisfying $Ff \circ \varepsilon_i^A = \varepsilon_i^B \circ f$ for $i = 1, 2$. If $A \leqslant B$, we may assume that $FA \leqslant FB$. F is a covariant functor from the category of all Boolean algebras and homomorphisms into itself. F is continuous, i.e., it commutes with formation of direct limits.

Let σ^A be the unique automorphism of FA satisfying $\sigma^A \circ \varepsilon_1^A = \varepsilon_2^A$ and $\sigma^A \circ \varepsilon_2^A = \varepsilon_1^A$. Then $(\sigma^A)^2 = \mathrm{id}_{FA}$, and $A \leqslant B$ implies $\sigma^A \subseteq \sigma^B$. Define the subalgebra SA of FA by

$$SA = \{x \in FA : \sigma^A(x) = x\}.$$

Clearly if $f : A \to B$ is a homomorphism then Ff maps SA into SB. Let

$$Sf = Ff \restriction SA.$$

Again, if $A \leqslant B$, then $SA \leqslant SB$ and the functor S is continuous. Note that $SA \leqslant_{\mathrm{rc}} FA$ since $\mathrm{pr}_{SA}^{FA}(x) = x \cdot \sigma^A(x)$ for $x \in FA$. In particular letting $E = S(\mathrm{Fr}\,\omega_2)$, $E \leqslant_{\mathrm{rc}} F(\mathrm{Fr}\,\omega_2)$ ($\cong \mathrm{Fr}\,\omega_2$). Ščepin showed that E is not projective. Using this result, we obtain the following claim.

Claim 2.4.1. $S(\mathrm{Fr}\,\omega_1) \not\leqslant_{\mathrm{proj}} F(\mathrm{Fr}\,\omega_1)$.

Proof. Suppose that $S(\mathrm{Fr}\,\omega_1) \leqslant_{\mathrm{proj}} F(\mathrm{Fr}\,\omega_1)$. $F(\mathrm{Fr}\,\omega_2)$ is the union of the continuous chain $(F(\mathrm{Fr}\,\omega_1 \cdot \alpha))_{\alpha<\omega_2}$. By continuity of the functor S, E is the union of the continuous chain $(S(\mathrm{Fr}\,\omega_1 \cdot \alpha))_{\alpha<\omega_2}$. Since $S(\mathrm{Fr}\,\omega_1 \cdot \alpha) \leqslant_{\mathrm{proj}} F(\mathrm{Fr}\,\omega_1 \cdot \alpha)$ for every $\alpha < \omega_2$ by the assumption, $S(\mathrm{Fr}\,\omega_1 \cdot \alpha) \leqslant_{\mathrm{proj}} S(\mathrm{Fr}\,\omega_1 \cdot \beta)$ for $\alpha < \beta < \omega_2$ by Lemma 1.5(2). By Theorem 1.3, it follows that E is projective. This is a contradiction. □ (Claim 2.4.1)

However, since $S(\mathrm{Fr}\,\omega_1) \leqslant_{\mathrm{rc}} F(\mathrm{Fr}\,\omega_1)$ and $|\mathrm{Fr}\,\omega_1| = \omega_1$, $S(\mathrm{Fr}\,\omega_1)$ is projective by Lemma 1.2(2). Hence $S(\mathrm{Fr}\,\omega_1)$ is free by Theorem 1.4. Thus we have two free

Boolean algebras $A = S(\mathrm{Fr}\,\omega_1)$, $B = F(\mathrm{Fr}\,\omega_1)$ of size ω_1 such that $A \leqslant_{\mathrm{rc}} B$ but $A \nleqslant_{\mathrm{proj}} B$. Let C be a countable Boolean algebra such that $C \mid A$. Then we have $C \leqslant_{\mathrm{rc}} B$. Since $|C| = \omega$, $|B| = \omega_1$ and B is free, it follows that $C \leqslant_{\mathrm{proj}} B$. Hence, by Lemma 1.5(1), we obtain that $C \mid B$. Thus A and B witness $F(\omega_1)$.

□ (Theorem 2.4)

We do not know whether $F(\kappa)$ holds for $\kappa \geqslant \omega_2$.

Corollary 2.5. (1) *There are 2^{ω_1} pairwise nonisomorphic $L_{\infty\omega_1}$-free Boolean algebras of size ω_1.*

(2) *There are 2^{ω_2} pairwise nonisomorphic $L_{\infty\omega_2}$-free Boolean algebras of size ω_2.*

(3) (V = L) *There are 2^{κ} pairwise nonisomorphic $L_{\infty\kappa}$-free Boolean algebras of size κ for every non-weakly compact cardinal κ.*

Proof. (1) Let S be a stationary and costationary subset of ω_1 consisting of limit ordinals and let $\{S_\alpha\}_{\alpha<\omega_1}$ be a partition of S such that S_α is stationary for every $\alpha < \omega_1$. For non-empty $X \in \mathcal{P}(\omega_1)$ let $S_X = \bigcup_{\alpha \in X} S_\alpha$. S_X is still stationary and costationary. Since $\mathrm{cof}(\alpha) = \omega$ for $\alpha \in S_X$ and $F(\omega)$ holds, by Proposition 2.3 there is $L_{\infty\omega_1}$-free Boolean algebra B_X of size ω_1 such that $\mathrm{Gd}(B_X) = (\omega_1 \setminus S_X)^{\sim}$. For each $X, Y \in \mathcal{P}(\omega_1)$, $X \neq Y$ we have that $\tilde{S}_X \neq \tilde{S}_Y$. It follows that B_X and B_Y are not isomorphic.

(2) Let S be a stationary subset of ω_2 consisting of ordinals of cofinality ω_1. S is non-reflecting and $F(\omega_1)$ holds, so by the same argument as in (1) we can construct 2^{ω_2} pairwise nonisomorphic $L_{\infty\omega_2}$-free Boolean algebras of size ω_2.

(3) Under V = L there is a non-reflecting stationary subset S of κ consisting of ordinals of cofinality ω (see e.g. [2]). Now the rest is the same as in (1). □

Note that by the same argument as in the proof of Lemma 23.7 in [8] we can show that each B_X constructed in (1) above satisfies the ccc. In Corollary 4.4(1) we shall show that there are maximally many $L_{\infty\omega_1}$-free Boolean algebras of size ω_1 with much stronger similarity to the free one.

In the rest of this section, we show that every $L_{\infty\omega_2}$-free Boolean algebra of size ω_2 is ω_2-free.

Lemma 2.6. *Let κ be a regular cardinal, A an $L_{\infty\kappa}$-free Boolean algebra of size κ, μ a regular cardinal $< \kappa$ and $(A_\alpha)_{\alpha<\kappa}$ an $L_{\infty\kappa}$-filtration of A. Then*

(1) *$A_\alpha \leqslant_{\mathrm{rc}} A_{\alpha+1}$ for all $\alpha < \kappa$ such that* $\mathrm{cof}(\alpha) \geqslant \omega_1$,

(2) *A_α is $L_{\infty\mu}$-free for all $\alpha < \kappa$ such that* $\mathrm{cof}(\alpha) = \mu$.

Proof. (1) Suppose that $\mathrm{cof}\,\alpha \geqslant \omega_1$ and $A_\alpha \nleqslant_{\mathrm{rc}} A_{\alpha+1}$. There exists an element a of $A_{\alpha+1}$ without the projection in A_α. By Zorn's lemma there exists a maximal increasing sequence $(a_\beta)_{\beta<\nu}$ of elements of A_α below a. Since $A_{\alpha+1}$ is free, A_α satisfies the ccc. Hence ν is a countable limit ordinal. Since $\mathrm{cof}\,\alpha \geqslant \omega_1$, there exists a $\mu < \alpha$ such that $\{a_\beta\}_{\beta<\nu} \subset A_{\mu+1}$. By the definition of $L_{\infty\kappa}$-filtrations, $A_{\mu+1} \mid A_{\alpha+1}$. Hence $p = \mathrm{pr}^{A_{\alpha+1}}_{A_{\mu+1}}(a)$ exists. Thus $p \in A_{\mu+1}$, $p \leqslant a$ and $b \leqslant p$ for all $b \in A_{\mu+1}$ with $b \leqslant a$. In particular we have $a_\beta \leqslant p$ for all $\beta < \nu$. This contradicts maximality of $(a_\beta)_{\beta<\nu}$.

(2) Suppose that $\operatorname{cof} \alpha = \mu$. Put

$$\mathcal{S} = \{C \in \mathrm{Sub}_{<\mu}(A_\alpha)\colon C \text{ is free and } C \mid A_{\beta+1} \text{ for some } \beta < \alpha\}.$$

We show that $\mathcal{S}$ is a Kueker system for A_α.

Claim 2.6.1. *$\mathcal{S}$ is cofinal in* $\mathrm{Sub}_{<\mu}(A_\alpha)$.

Proof. Suppose that $D \in \mathrm{Sub}_{<\mu}(A_\alpha)$. Since $\operatorname{cof} \alpha = \mu$, there exists $\beta < \alpha$ such that $D \subset A_{\beta+1}$. So there is a free C such that $|C| < \mu$, $D \subset C$ and $C \mid A_{\beta+1}$. □ (Claim 2.6.1)

Claim 2.6.2. *For any $\mathcal{S}' \subset \mathcal{S}$ with $|\mathcal{S}'| < \mu$ there exists $F \in \mathcal{S}$ such that $C \mid F$ for all $C \in \mathcal{S}'$.*

Proof. Let $D = \langle \bigcup_{C\in\mathcal{S}'} C\rangle$. Then $|D| < \mu$. For $C \in \mathcal{S}'$ put

$$\beta_C = \min\{\beta + 1\colon C \mid A_{\beta+1}\}.$$

There exists $\beta < \alpha$ such that $A_{\beta+1} \supset D$ and $\beta_C < \beta$ for every $C \in \mathcal{S}'$. Since $A_{\beta_C} \mid A_{\beta+1}$, we have $\forall C \in \mathcal{S}'$ $[C \mid A_{\beta+1}]$. Hence there exists $F \subset A_{\beta+1}$ such that $|F| < \mu$, $F \mid A_{\beta+1}$ and $\forall C \in \mathcal{S}'$ $[C \mid F]$. □ (Claim 2.6.2) □ (Lemma 2.6)

Since every countable atomless Boolean algebra is free, every atomless (i.e., $L_{\infty\omega}$-free) Boolean algebra is ω_1-free. For ω_2, we still have the similar situation:

Theorem 2.7. *Every $L_{\infty\omega_2}$-free Boolean algebra of size ω_2 is ω_2-free.*

Proof. Let A be an $L_{\infty\omega_2}$-free Boolean algebra of size ω_2 and $(A_\alpha)_{\alpha<\omega_2}$ an $L_{\infty\omega_2}$-filtration of A. By the definition of $L_{\infty\omega_2}$-filtrations it can be easily seen that, if $\operatorname{cof} \alpha \leqslant \omega$, then A_α is free. Suppose that $\operatorname{cof} \alpha = \omega_1$. By Lemma 2.6, $A_\alpha \leqslant_{\mathrm{rc}} A_{\alpha+1}$ and A_α is $L_{\infty\omega_1}$-free. Since $|A_\alpha| = \omega_1$ and $A_{\alpha+1}$ is free, A_α is projective by Lemma 1.2(2). Thus A_α is a projective $L_{\infty\omega_1}$-free Boolean algebra of size ω_1. So by Theorem 1.4 it is free. Hence A_α is free for all $\alpha < \omega_2$ and A is ω_2-free by definition. □

The converse of Theorem 2.7 is not true; for example, let $B = \mathrm{Fr}\,\kappa \times \mathrm{Fr}\,\kappa^+$ for an infinite cardinal κ. Then B is κ^+-free, since $\mathrm{Fr}\,\kappa^+$ is the union of the continuous chain of $(\mathrm{Fr}\,\kappa\cdot\alpha)_{\alpha<\kappa^+}$. B is projective but non-free. Hence B is not $L_{\infty\kappa^+}$-free by Theorem 1.4.

For $\kappa > \omega_2$ we still do not know if there is an $L_{\infty\kappa}$-free non-κ-free Boolean algebra of size κ. This is also connected with the open problem whether every $L_{\infty\kappa}$-free Boolean algebra of size κ for weakly compact κ is free:

Theorem 2.8. *Let κ be a weakly compact cardinal. Then every κ-free Boolean algebra of size κ is free.*

Proof. The proof is just like that of Theorem 3.2 in Chapter IV of [5].

Suppose that there is a κ-free Boolean algebra B of size κ which is not free. By Theorem 1.4, B is not projective. Since B is κ-free, there is a filtration $(B_\alpha)_{\alpha<\kappa}$ of B such that $B_0 = 2$ and $E_\alpha \cong \mathrm{Fr}\,|\alpha + \omega|$ for $0 < \alpha < \kappa$. By Proposition 2.2,

$$E = \{\nu < \kappa \colon B_\nu \not\leqslant_{\mathrm{wproj}} B\}$$

is a stationary subset of κ. In particular $B_\nu \not\leqslant_{\mathrm{proj}} B$ for $\nu \in E$. Thus for each $\upsilon \in E$

$$S_\nu = \{\mu < \kappa \colon \nu < \mu,\ B_\nu \not\leqslant_{\mathrm{proj}} B_\mu\}$$

is stationary. For $\nu \in \kappa \backslash E$, let $S_\nu = E$. By weak compactness of κ, there is a regular $\lambda < \kappa$ such that $S_\nu \cap \lambda$ is a stationary subset of λ for every $\nu < \lambda$. Again by Proposition 2.2, it follows that B_λ is not projective hence non-free. But this is a contradiction to the choice of the B_α's. $\square$

If we assume $\diamondsuit_{\omega_1}$, we can construct maximally many ccc $L_{\infty\omega_1}$-free Boolean algebras of size ω_1 which do not have any finitely additive strictly positive measure, by a similar argument as above. Under $\mathrm{V} = \mathrm{L}$, this is also true for any regular non-weakly compact uncountable κ in place of ω_1.

On the other hand, if we assume $\mathrm{MA} + \neg\mathrm{CH}$ there are no such Boolean algebras in ω_1, since, under MA, every ccc Boolean algebra of size $< 2^\omega$ is embeddable in $\mathscr{P}(\omega)$.

3. Boolean algebras which are embeddable into $L_{\infty\omega_1}$-free Boolean algebras

In this section, we give a partial answer to our third question: we show that a pseudo-tree algebra is embeddable into an $L_{\infty\omega_1}$-free Boolean algebra if and only if it satisfies the condition (Stab) defined in the Introduction.

Let O be a given ordered set and O' a subset of O. For $x \in O$ let us write

$$O'\!\uparrow\! x = \{y \in O' \colon y \geqslant x\},$$

$$O'\!\upuparrows\! x = \{y \in O' \colon y > x\},$$

$$O'\!\downarrow\! x = \{y \in O' \colon y \leqslant x\},$$

$$O'\!\downdownarrows\! x = \{y \in O' \colon y < x\}.$$

Similarly, for a subset X of O, we define

$$O'\uparrow X = \{y \in O' : y \geqslant x \text{ for all } x \in X\},$$
$$O'\downarrow X = \{y \in O' : y \leqslant x \text{ for all } x \in X\} \quad \text{etc.}$$

The next lemma follows from Theorem 1.1(2) and Lemma 1.5(1):

Lemma 3.1. *For a Boolean algebra B if there is a set $\mathcal{S}$ of countable subalgebras such that*

$\mathcal{S}$ is cofinal in $\mathrm{Sub}_{<\omega_1}(B)$ *and each $C \in \mathcal{S}$ is atomless and relatively complete in B*

then $B \oplus \mathrm{Fr}\,\omega_1$ is $L_{\infty\omega_1}$-free.

In particular B is then embeddable in an $L_{\infty\omega_1}$-free Boolean algebra. □

A partially ordered set T is a pseudo-tree if the set $T\downarrow t$ is linearly ordered for every $t \in T$. A branch in T is a maximal linearly ordered subset of T. For a pseudo-tree T the pseudo-tree algebra Treealg(T) over T is the subalgebra of $\mathcal{P}(T)$ generated by $\{T\uparrow t : t \in T\}$. We say that a Boolean algebra B is a pseudo-tree algebra if B is of the form Treealg(T) for some pseudo-tree T. The notion of pseudo-tree algebra has been studied in [11]. If T is linearly ordered, Treealg(T) is also called an interval algebra over T.

As for a tree algebra, each element a of a pseudo-tree algebra Treealg(T) can be represented in a normal form (see [9]). We shall call the elements in T which appear in such a normal form representation of a the end-points of a.

An element t in a pseudo-tree T is said to be branched if t is not maximal in T and there are branches b and b' of T such that $b \neq b'$ and $b \cap b' = T\downarrow t$. An initial segment u of a pseudo-tree T is said to be branched if u is not a branch of T and there are branches b and b' of T such that $b \neq b'$ and $b \cap b' = u$. Note that if u is not branched then $T\uparrow u$ is downward directed.

The cofinality of an upward directed partially ordered set O is the minimal cardinal κ such that there is a cofinal subset of O of order type κ. The coinitiality of a downward directed partially ordered set O is the minimal cardinal κ such that there is a coinitial subset of O of order type κ^*.

A gap in a partially ordered set O is a pair (X, Y) of subsets of O such that X is upward directed, Y is downward directed and $X \leqslant Y$. A gap (X, Y) has the cofinality type (κ, λ^*) if X has cofinality κ and Y has coinitiality λ. A gap (X, Y) in O is unfilled if there is no $x \in O$ such that $X \leqslant x \leqslant Y$.

If X is a subset of a pseudo-tree T and $t \in T$, the quantifier-free type of t over X is decided by the pair $(X\downarrow t, X\uparrow t)$. We shall call this pair simply the type of t over X or the type over X realized by t etc.

In the following lemmas we show that a pseudo-tree with (Stab) can be embedded in another pseudo-tree which looks like a normal binary tree and which still satisfies (Stab).

Lemma 3.2. *Let T_0 be a pseudo-tree which satisfies* (Stab). *Then there exists a pseudo-tree $T \supseteq T_0$ such that T satisfies* (Stab) *and the following* (a)–(c).

(a) *For every $t, t' \in T$ if $(T{\downarrow}t) \cap (T{\downarrow}t') \neq \emptyset$ then there is the maximal element in $(T{\downarrow}t) \cap (T{\downarrow}t')$ — we shall denote this element by* $\mathrm{bra}_T(t, t')$.

(b) *For every $t \in T$ if t is branched in T there are $t', t'' \in T$ such that $t < t', t''$ and for every $u \in T{\uparrow\uparrow}t$ we have either $t' \leqslant u$ or $t'' \leqslant u$.*

(c) *Every unfilled gap in T has the cofinality type (ω, ω^*).*

Proof. Let $T_0 = (T_0, \leqslant_{T_0})$. Let I be the set of all linearly ordered initial segments in T_0. For each $s \in I$ we insert some new elements X_s between s and $T_0{\uparrow\uparrow}s$ and define an ordering $\leqslant_s$ on $T_0 \cup X_s$. The pseudo-tree algebra we look for is then defined as $T = T_0 \cup \bigcup_{s \in I} X_s$ with the ordering generated from $\leqslant_s$, $s \in I$. Now let $s \in T$.

Case 1: *s is branched.* Let $Y \subseteq T_0$ be a maximal subset of T with the property that if $y, y' \in Y$ and $y \neq y'$ then $(T_0{\downarrow}y) \cap (T_0{\downarrow}y') = s$. Let $\{t_\alpha : 0 < \alpha \leqslant \kappa\}$ be a 1–1 enumeration of Y where $\kappa = |Y|$. Let $X_s = \{u^s_\alpha : 0 < \alpha < \kappa\} \cup \{v^s_\alpha : 0 < \alpha \leqslant \kappa\}$, where the u^s_α's and v^s_α's are distinct new elements. Let $\leqslant_s$ be the partial ordering on $T_0 \cup X_s$ generated by $\leqslant_{T_0}$ and the relation defined by the following inequalities:

(1) $x \leqslant u^s_1$, for all $x \in s$,
(2) $u^s_\alpha \leqslant u^s_\beta$, for all $0 < \alpha \leqslant \beta < \kappa$,
(3) $u^s_\alpha \leqslant v^s_\beta$, for all $0 < \alpha < \kappa$ and $\alpha \leqslant \beta \leqslant \kappa$,
(4) $v^s_\alpha \leqslant x$, for all $x \in T_0{\uparrow}s$ such that $(T_0{\downarrow}x) \cap (T_0{\downarrow}t_\alpha) \supset s$.

Case 2: *s is not branched.* If $T_0{\uparrow}s \neq 0$ and either s is of uncountable cofinality or $T_0{\uparrow}s$ is of uncountable coinitiality, let $X_s = \{x^s\}$ and $s \leqslant x^s \leqslant T_0{\uparrow}s$. Otherwise let $X_s = \emptyset$ (and $\leqslant_s = \leqslant_{T_0}$). It is easily seen that T satisfies (a)–(c).

Claim 3.2.1. *T satisfies* (Stab).

Proof. Let us assume by way of contradiction that there are a countable $X \subseteq T$ and an uncountable $Y \subseteq T$ such that the elements of Y realize distinct types over X. Without loss of generality let $X \cap Y = \emptyset$. If $\{y \in Y : (T{\uparrow}y) \cap X = \emptyset\}$ is uncountable we may assume without loss of generality that $T{\uparrow}y \cap X = \emptyset$ for every $y \in Y$. Now by the usual binary tree argument we may also assume without loss of generality that X is order isomorphic to ${}^{\omega>}2$ and for every $x, x' \in X$ with $x \neq x'$ we have $T_0{\downarrow}x \neq T_0{\downarrow}x'$. For each $x, x' \in X$ with $x < x'$ let $t_{x,x'} \in T_0$ be such that $x \leqslant t_{x,x'} \leqslant x'$. Let $X_0 = \{t_{x,x'} : x, x' \in X, x < x'\}$. For each $y \in Y$ let $t_y \in T_0{\uparrow}y$. Then t_y, $y \in Y$ realize distinct types over X_0. This is a contradiction to the assumption that T_0 satisfies (Stab).

If $\{y \in Y : (T{\uparrow}y) \cap X = \emptyset\}$ is countable we may assume without loss of generality that there is $x_0 \in X$ such that $y < x_0$ for all $y \in Y$ and $X \subseteq T{\downarrow}x_0$. Without loss of generality we may also assume that for every $x, x' \in X$ with $x < x'$

there exists $t \in T_0$ such that $x \leq t \leq x'$. For every $x, x' \in X$ with $x < x'$ let $t_{x,x'} \in T_0$ be such that $x \leq t_{x,x'} \leq x'$. Let $X_0 = \{t_{x,x'}: x, x' \in X, x < x'\}$. Without loss of generality elements of Y realize distinct types over X_0. Since T_0 satisfies (Stab), we may also assume that $Y \subseteq T \setminus T_0$.

Case 1: *Uncountably many $y \in Y$ are of the form u^s_α or v^s_α in Case* 1 *of the construction of T.* We may then assume without loss of generality that every $y \in T$ is of this form. For each $y \in Y$ if $y = u^s_\alpha$ or $y = v^s_\alpha$ for some $s \in I$, let $y^* = v^s_{\alpha'}$ for some $\alpha' \neq \alpha$ such that $v^s_{\alpha'} \nleq x_0$ and let $t_y \in T_0 \uparrow y^*$. Then t_y, $y \in Y$ realize distinct types over X_0. This is a contradiction to the assumption that T_0 satisfies (Stab).

Case 2: *Only countably many $y \in Y$ are of the form u^s_α or v^s_α in Case* 1 *of the construction of T.* In this case we may assume without loss of generality that every $y \in Y$ is of the form x^s, as in Case 2 of the construction of T. Then for each $y \in Y$ either $T_0 \downdownarrows y$ is of uncountable cofinality or $T_0 \upuparrows y$ is of uncountable coinitiality. In both cases we can find $t_y \in T_0$ such that $X_0 \downarrow y \leq t_y \leq X_0 \uparrow y$. Clearly t_y, $y \in Y$ realize distinct types over X_0. Again this is a contradiction to the fact that T_0 satisfies (Stab). $\square$ (Claim 3.2.1) $\square$ (Lemma 3.2)

Lemma 3.3. *Let T_0 be a pseudo-tree such that T_0 satisfies* (Stab), (a)–(c). *Then there exists a pseudo-tree $T \supseteq T_0$ such that T satisfies* (Stab), (a)–(c) *and*

(d) *For every $t \in T$, if t is not branched and $T \upuparrows t$ has no minimal element then $T \upuparrows t$ has uncountable coinitiality.*

Proof. Let T be the pseudo-tree obtained from T_0 by replacing every non-branched $t \in T_0$ such that $T_0 \upuparrows t$ has uncountable coinitiality by an ordering of order type ω. Note that with this construction we add only unfilled gaps of cofinality type (ω, ω^*). $\square$

Lemma 3.4. *Let T be a pseudo-tree such that T satisfies* (Stab), (a)–(c). *Then for every countable $X \subseteq T$ there exists a countable $Y \subseteq T$ such that $X \subseteq Y$, Y is closed with respect to* $\mathrm{bra}_T(\cdot, \cdot)$ *and for every $t, t' \in T$, $(Y \downarrow t) \cap (Y \uparrow t')$ has a minimal and a maximal element provided that $(Y \downarrow t) \cap (Y \uparrow t') \neq \emptyset$.*

Proof. Let Y' be the closure of Y in T with respect to $\mathrm{bra}_T(\cdot, \cdot)$. Clearly Y' is still countable. Let $\mathscr{D}$ be a maximal set of branches in Y' with the property that for every $b \in \mathscr{D}$, $T \downarrow b \neq \emptyset$ and for $b, b' \in \mathscr{D}$ if $b \neq b'$ then $(T \downarrow b) \cap (T \downarrow b') = \emptyset$. Since T satisfies (Stab), $\mathscr{D}$ is countable. For each $b \in \mathscr{D}$ let $d_b \in T \downarrow b$. Now let $\mathscr{U}$ be the set of all branches b in Y' such that $T \uparrow b \neq \emptyset$. T satisfies (Stab) and (a), so that $\mathscr{U}$ is countable. For each $b \in \mathscr{U}$ let $u_b \in T \downarrow b$. Let $Y'' = Y' \cup \{d_b: b \in \mathscr{D}\} \cup \{u_b: b \in \mathscr{U}\}$. Since T satisfies (Stab), we can enumerate all the types (D, U) in T over Y'' such that $D \neq \emptyset$ and $U \neq \emptyset$ by (D_n, U_n), $n \in \omega$. Let $X_n = \{t \in T: D_n \leq t \leq u_n\}$. By condition (c), for every $n \in \omega$ there exist $k_n, l_n \in \{0, \omega\}$ and a subset V_n of X_n with order type $k_n^* + l_n$ such that V_n is cofinal and coinitial in X_n. Let $Y = Y'' \cup \bigcup_{n \in \omega} V_n$. $\square$

Theorem 3.5. *Let T be a pseudo-tree and B the pseudo-tree algebra over T. Then B is embeddable in an $L_{\infty\omega_1}$-free Boolean algebra iff B satisfies* (Stab) *iff T satisfies* (Stab).

Proof. If B satisfies (Stab) then clearly T also satisfies (Stab). Suppose that T satisfies (Stab). By Lemmas 3.2, 3.3 and 3.4 we may assume without loss of generality that T satisfies (a)–(d) as well. For $X \subseteq T$, let

$$B_X = \{a \in \mathrm{Treealg}(T)\text{: end-points of } a \text{ are in } X\}$$

and

$$S = \{B_X : X \subseteq T,\ X \text{ is countable and closed with respect to } \mathrm{bra}_T(\cdot,\cdot), \\ \text{for every } t, t' \in T,\ (X{\downarrow}t) \cap (X{\uparrow}t') \text{ has a minimal and a} \\ \text{maximal element provided that } (X{\downarrow}t) \cap (X{\uparrow}t') \neq \emptyset\}.$$

By Lemma 3.4, S is cofinal in $[B]^{\aleph_0}$. So by Lemma 3.1 we are done by the following claim.

Claim 3.5.1. *The elements of S are relatively complete subalgebras of* Treealg(T).

Proof. Let $B \in S$, say $B = B_X$ for some $X \subseteq T$ as in the definition of S. It is enough to show that every $a \in \mathrm{Treealg}(T)$ has a projection in B_X. For the simplicity let us consider the case that $a = (T{\uparrow}t_0)\setminus(T{\uparrow}t_1)$ for some $t_0, t_1 \in T$, $t_0 < t_1$. The general case can be proved similarly.

If there is no $x \in X$ such that $t_0 \leqslant x \leqslant t_1$ then $\emptyset$ is the projection of a in B_X. Otherwise let x_0 be the minimal element of X such that $t_0 \leqslant x_0 \leqslant t_1$. If there is no $x \in X$ such that $x_0 < x \leqslant t_1$ then $\emptyset$ is again the projection of a in B_X. Otherwise let x_1 be the maximal element of X such that $x_0 < x_1 \leqslant t_1$.

Case 1: *x_1 is branched.* We shall define $a', a'' \in B_X$ so that $(T{\uparrow}x_0)\setminus(T{\uparrow}x_1) \cup a' \cup a''$ will be the projection of a in B_X. Let t' and t'' be as in (b) for $t = x_1$. Assume without loss of generality $t'' \leqslant t_1$. If $X{\uparrow}t' \neq \emptyset$ let x' be the minimal element in $X{\uparrow}t'$. The uniqueness of x' follows from the fact that X is closed with respect to $\mathrm{bra}_T(\cdot,\cdot)$. Let $a' = T{\uparrow}x'$. If $X{\uparrow}t' = \emptyset$ let $a' = \emptyset$. Now if $t_1 = t''$ let $a'' = \emptyset$. Otherwise we have $t'' < t_1$. If $(X{\uparrow}t'')\setminus(X{\uparrow}t_1) = \emptyset$ let $a'' = \emptyset$. Otherwise, as above, there is the unique minimal element $x'' \in (X{\uparrow}t'')\setminus(X{\uparrow}t_1)$. Let $a'' = T{\uparrow}x''$.

Case 2: *x_1 is not branched and there is a minimal element $t \in T{\uparrow\uparrow}x_1$.* If $t = t_1$ or $(X{\uparrow}t)\setminus(X{\uparrow}t_1) = \emptyset$ then $(T{\uparrow}x_0)\setminus(T{\uparrow}x_1)$ is the projection of a in B_X. Otherwise there is a unique minimal element x in $(X{\uparrow}t)\setminus(X{\uparrow}t_1)$. $((T{\uparrow}x_0)\setminus(T{\uparrow}x_1)) \cup (T{\uparrow}x)$ is then the projection of a in B_X.

Case 3: *x_1 is not branched and there is no minimal element in $T{\uparrow\uparrow}x_1$.* By (d), $T{\uparrow\uparrow}x_1$ has cofinality $\geqslant \omega_1$. Since X is countable there is $t \in T{\uparrow\uparrow}x_1$ such that $X{\uparrow\uparrow}x_1 \subseteq T{\uparrow}t$. If $(X{\uparrow}t)\setminus(X{\uparrow}t_1) = \emptyset$ then $(T{\uparrow}x_0)\setminus(T{\uparrow}x_1)$ is the projection of a in B_X. Otherwise let x be the minimal element in $(X{\uparrow}t)\setminus(X{\uparrow}t_1)$. Then $((T{\uparrow}x_0)\setminus(T{\uparrow}x_1)) \cup (T{\uparrow}x)$ is the projection of a in B_X. □ (Claim 3.5.1) □ (Theorem 3.5)

Since every ordered set, in particular every cardinal κ satisfies (Stab) and linearly ordered sets are pseudo-trees, we obtain the following corollary.

Corollary 3.6. *For any cardinal κ there is an $L_{\infty\omega_1}$-free Boolean algebra B which does not satisfy the κ-cc.* $\square$

In contrast to this result, every $L_{\infty\omega_2}$-free Boolean algebra has precalibre $\aleph_1$ (hence satisfies the ccc). We do not know if every $L_{\infty\omega_2}$-free Boolean algebra is absolutely ccc. However, unless there is some large cardinal, every $L_{\infty\omega_2}$-free Boolean algebra still has precalibre $\aleph_1$ in any generic extension which preserves cardinals:

Proposition 3.7. *Assume that the covering lemma holds. Then any $L_{\infty\omega_2}$-free Boolean algebra remains of precalibre $\aleph_1$ in any generic extension in which cardinals are preserved.*

Proof. If the covering lemma holds in the ground model M then it still holds in any generic extension of M. Suppose that B is an $L_{\infty\omega_2}$-free Boolean algebra in M and P a poset preserving cardinals. If X is a subset of B of cardinality $\aleph_1$ in M^P then by the covering lemma there is a $Y \subseteq B$, in M, of cardinality $\aleph_1$ including X. Since B is $L_{\infty\omega_2}$-free, there is a free subalgebra C of B in M including Y. Since C remains free in M^P it follows that in M^P there exists an uncountable centred subset of X. $\square$

This problem is also connected with Question 2 of [17].

Since every ω_1-tree satisfies (Stab) and trees are pseudo-trees, we obtain the following corollary of Theorem 3.5.

Corollary 3.8. *Every ω_1-tree T can be embedded into an $L_{\infty\omega_1}$-free Boolean algebra.* $\square$

We can prove the corollary above directly from the fact that $\mathrm{Treealg}(T) \oplus \mathrm{Fr}\,\omega_1$ is already $L_{\infty\omega_1}$-free. In particular if T is a Suslin tree we obtain by this construction an $L_{\infty\omega_1}$-free Boolean algebra which satisfies the ccc but is not absolutely ccc.

Of course we cannot embed everything in some $L_{\infty\omega_1}$-free Boolean algebra. For example it is clear that $\mathbb{R}$ does not satisfy the condition (Stab). In particular $\mathbb{R}$ is not embeddable in any $L_{\infty\omega_1}$-free Boolean algebra.

4. κ-potentially free Boolean algebras

In this section we give a characterization of κ-potentially free Boolean algebras of size κ and construct κ-potentially free non-free Boolean algebras of size κ.

Let κ be a regular cardinal and S a subset of κ. As in [1] we say that S is fat if for every club subset C of κ and every $\beta<\kappa$, $S\cap C$ contains a closed set of ordinals of order type β. Note that $S\subseteq\omega_1$ is fat if it is stationary. We say that $\mathrm{Gd}(B)$ is fat if $\mathrm{Gd}((B_\alpha)_{\alpha<\kappa})$ is fat for some/any filtration $(B_\alpha)_{\alpha<\kappa}$. We need the following theorem in [1]:

Theorem 4.1 (Stavi, Shelah). *Let κ be a strongly inaccessible cardinal or the successor of a cardinal λ where λ is regular and $\lambda=\lambda^{<\lambda}$, or λ is singular strong limit. Let $S\subset\kappa$ be fat. Then there exists a $(<\kappa,\infty)$-distributive complete Boolean algebra C such that*

$$[\![\check{S} \text{ contains a club in } \check{\kappa}]\!]^{(C)}=1. \qquad \square$$

Theorem 4.2. *Let κ be regular and B a Boolean algebra of size κ.*

(1) *If b is κ-potentially free, then B is $L_{\infty\kappa}$-free and* $\mathrm{Gd}(B)$ *is fat.*

(2) *Let κ satisfy the condition in Theorem* 4.1. *If B is $L_{\infty\kappa}$-free and* $\mathrm{Gd}(B)$ *is fat, then B is κ-potentially free.*

Proof. (1) Suppose that B is κ-potentially free. Fix a $(<\kappa,\infty)$-distributive complete Boolean algebra C such that $[\![\check{B}\cong\mathrm{Fr}\,\check{\kappa}]\!]^{(C)}=1$. Since the satisfaction of $L_{\infty\kappa}$-sentences is absolute under this Boolean extension, B is $L_{\infty\kappa}$-free. Let $(B_\alpha)_{\alpha<\kappa}$ be a $L_{\infty\kappa}$-filtration of B. Note that

$$B_\alpha\leqslant_{\mathrm{wproj}}B \iff B_\alpha\leqslant_{\mathrm{proj}}B_{\alpha+1}$$

for $\alpha<$ by Proposition 1.7.

Since C is $(<\kappa,\infty)$-distributive and $|B_\alpha|\leqslant|B_{\alpha+1}|<\kappa$,

$$\begin{aligned} B_\alpha\leqslant_{\mathrm{wproj}}B &\iff B_\alpha\leqslant_{\mathrm{proj}}B_{\alpha+1}\\ &\iff [\![\check{B}_\alpha\leqslant_{\mathrm{proj}}\check{B}_{\alpha+1}]\!]^{(C)}=1\\ &\iff [\![\check{B}_\alpha\leqslant_{\mathrm{wproj}}\check{B}]\!]^{(C)}=1.\end{aligned}$$

Hence by Proposition 2.2,

$$[\![(\mathrm{Gd}((B_\alpha)_{\alpha<\kappa}))^\vee=\mathrm{Gd}((B_\alpha)^\vee_{\alpha<\kappa}) \text{ and } \mathrm{Gd}((B_\alpha)^\vee_{\alpha<\kappa}) \text{ contains a club set}]\!]^{(C)}=1.$$

Since C is $(<\kappa,\infty)$-distributive it follows that $\mathrm{Gd}((B_\alpha)_{\alpha<\kappa})$ is fat.

(2) Assume that B is $L_{\infty\kappa}$-free and $\mathrm{Gd}(B)$ is fat. By Theorem 4.1 there exists a complete $(<\kappa,\infty)$-distributive Boolean algebra C such that

$$[\![(\mathrm{Gd}(B))^\vee \text{ contains a club set in } \check{\kappa}]\!]^{(C)}=1.$$

By Proposition 2.2, it follows that

$$[\![\check{B} \text{ is projective}]\!]^{(C)}=1.$$

On the other hand, since C is $(<\kappa,\infty)$-distributive and B is $L_{\infty\kappa}$-free,

$$[\![\check{B} \text{ is } L_{\infty\check{\kappa}}\text{-free}]\!]^{(C)}=1.$$

Hence, by Theorem 1.4, it follows that

$$[\![\check{B} \cong \mathrm{Fr}\,\check{\kappa}\,]\!]^{(C)} = 1. \qquad \square$$

Corollary 4.3. *Let B be a Boolean algebra of size ω_1. Then the following are equivalent:*

(1) *B is ω_1-potentially free.*

(2) *B is $L_{\infty\omega_1}$-free and* $\mathrm{Gd}(B) \neq \bar{\emptyset}$.

(3) *There is a complete Boolean algebra C such that*

$$[\![\check{\omega}_1 = \omega_1 \text{ and } B \text{ is free}]\!]^{(C)} = 1.$$

Proof. The equivalence of (1) and (2) follows from Theorem 4.2.

(1)$\Rightarrow$(3) is clear.

(3)$\Rightarrow$(2): Let C be as in (3) and let $(B_\alpha)_{\alpha<\omega_1}$ be a filtration of B. Then

$$[\![\mathrm{Gd}((B_\alpha)_{\alpha<\omega_1}) \text{ contains a club set in } \omega_1]\!]^{(C)} = 1.$$

It follows that $\mathrm{Gd}((B_\alpha)_{\alpha<\omega_1})$ is stationary.

By

$$[\![B \text{ is free}]\!]^{(C)} = 1,$$

it follows that $\{\alpha < \omega_1 \colon B_\alpha \leqslant_{\mathrm{rc}} B\}$ is unbounded in ω_1 and for every $\alpha < \omega_1$ there exists $b \in B$ which is independent from B_α. Hence by Theorem 1.4 we can find a subsequence $(C_\alpha)_{\alpha<\omega_1}$ of $(B_\alpha)_{\alpha<\omega_1}$ such that $C_\alpha \mid C_\beta$ for every $\alpha < \beta < \omega_1$. By Theorem 1.1 it follows that B is $L_{\infty\omega_1}$-free. $\square$

Corollary 4.4. (1) *There are 2^{ω_1} pairwise non-isomorphic ω_1-potentially free Boolean algebras of size ω_1.*

(2) (CH) *There are 2^{ω_2} pairwise non-isomorphic ω_2-potentially free Boolean algebras of size ω_2.*

(3) (V = L) *There are 2^κ pairwise non-isomorphic κ-potentially free Boolean algebras of size κ for every non-weakly compact regular uncountable cardinal κ.*

Proof. For $X \in \mathcal{P}(\kappa)$ let S_X be the non-reflecting stationary subset of κ as in the proof of Corollary 2.5. By Lemma 1.1 in [1], $\kappa \backslash S_X$ is a fat subset of κ for each $X \in \mathcal{P}(\kappa)$. By CH in (2) and V = L in (3), we can apply Theorem 4.2 to the B_X's defined as in the proof of Corollary 2.5, and obtain maximally many κ-potentially free Boolean algebras. $\square$

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